Math 206B Lecture 25 Notes

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1 Inequalities in Algebraic Combinatorics

1.1 Largest number of tableau for a partition of n

Proposition 1.1. $(f^\lambda)^2 \leq n!$

Proof. This is because $\sum_{|\lambda|=n} (f^\lambda)^2 = n!$. □

Here is a restatement of this fact:

Corollary 1.1. $|\operatorname{SYT}(\lambda)|^2 \leq n!$.

This seems much less obvious, and requires RSK to prove it directly.

Corollary 1.2. *Denote $D(n) = \max_{|\lambda|=n} f^\lambda$. Then $D(n) \geq \sqrt{n!/p(n)}$, where $p(n)$ is the number of partitions of n.*

Here is a conjecture:

Theorem 1.1. *The number of $|\lambda| = n$ such that $f^\lambda = D(n)$ is $O(1)$.*

This is open, even though it seems like it should be obvious. In fact, we don't know if it is $e^{O(\sqrt{n})}$. The following, however, is known.

Theorem 1.2 (V-K)**.** $D(n) < \sqrt{n!}\alpha^{\sqrt{n}}$ *for some* $\alpha > 1$.

So we have this upper bound and the lower bound $\sqrt{n!}/\beta^{\sqrt{n}}$. Here is a conjecture that Professor Pak wants to prove:

Theorem 1.3. *The following limit exists:*

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \left(\frac{D(n)}{\sqrt{n!}} \right).$$

In 1954, someone at Los Alamos, used extra computing power to compute character tables of S_n for $n \leq 15$. They became interested in $D(n)$ and conjectured that $D)n_{\leq}\sqrt{n!}/n$. This was proven false about 15 years later.

Theorem 1.4 (Bufetov)**.** *Let $H(n) = 1/p(n) \sum_{|\lambda|=n} f^\lambda$. Then the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\left(\frac{H(n)}{\sqrt{n!}}\right).$$

Theorem 1.5 (V-K)**.** *Let $f^\lambda = D(n)$. Then the shape of λ (scaled by $\sqrt{n}$) looks like a rotated version of the graph*

$$\Phi(x) = \frac{2}{\pi}(x \arcsin(x/\sqrt{2}) + \sqrt{2-x}).$$

Corollary 1.3. *The average longest increasing subsequence of a permutation is $2\sqrt{n}$.*

1.2 Bounds for Littlewood-Richardson coefficients

Theorem 1.6 (PPY)**.** $(c^\lambda_{\mu,\nu})^2 \le \binom{n}{k}$.

The upper bound is actually somewhat tight: the idea is to show that

$$\sum_{|\lambda|=n} \sum_{|\mu|=k, |\nu|=n-k} (c^\lambda_{\mu,\nu})^2 = \sum_{\alpha\in \mathrm{conj}(H=S_k\times S_{n-k})} \frac{z_\alpha(S_n)}{z_\alpha(H)} \alpha \overset{\geq}{=} 1 \binom{n}{k}.$$

Then

$$\max_{\lambda,\mu,\nu} c^\lambda_{\mu,\nu} \ge \frac{\sqrt{\binom{n}{k}}}{\sqrt{p(k)}p(n-k)p(n)}.$$

Proof. The idea of the proof of the theorem is to show that $\binom{n}{k} f^\mu f^\nu$ is the dimension of $\mathrm{ind}^{S_n}_{S_k\times S_{n-k}} S^\mu \otimes S^\nu$ and decompose the representation into irreducible repesentations. Then

$$\sum_{|\mu|=k} \sum_{|\nu|=n-k} c^\lambda_{\mu,\nu} f^\mu f^\nu = f^\lambda,$$

and

$$\sum_{|\lambda|=n} (c^\lambda_{\mu,\nu})^2 \le \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} \frac{f^\lambda}{f^\mu f^\nu} = \frac{1}{f^\mu f^\nu} f^\mu f^\nu \binom{n}{k} = \binom{n}{k}.$$

So $(c^\lambda_{\mu,\nu})^2 \le \binom{n}{k}$. □

Theorem 1.7 (PPY, 2018)**.** *There exist $\lambda, \mu\,\nu$ such that $c^\lambda_{\mu,\nu} = 2^n / e^{O(-\sqrt{n})}$.*

1.3 Bounds on the number of skew tableau of size n

Let $f^{\lambda\backslash\mu} = |\,\mathrm{SYT}(\lambda \setminus \mu)|$. We know that this number is the determinant of a matrix we get from λ, μ. Can we understand this number better? Our previous considerations give us the following:

Proposition 1.2. *Let $|\lambda| = n$ and $|\mu| = k$. Then*

$$f^{\lambda\backslash\mu} \leq \sqrt{\binom{n}{k} p(n-k) \sqrt{(n-k)!}}.$$

Proof. This follows from the previous inequalities applied to the identity:

$$f^{\lambda\backslash\mu} = \sum_{|\nu|=n-k} c^{\lambda}_{\mu,\nu} f^{\nu}.$$

□

What about lower bounds?

Theorem 1.8 (Naruse,MPP)**.**

$$f^{\lambda\backslash\mu} = n! \sum_{D\in\mathcal{E}(\lambda\backslash\mu)} \prod_{(i,j)\notin D} \frac{1}{h_{i,j}},$$

where $\mathcal{E}(\lambda \setminus \mu)$ is the set of "excited diagrams" (start with chips in the removed shape μ, and move them to the right or down to get a configuration in $\lambda \setminus \mu$).

Example 1.1. Suppose $\lambda = (3,3)$ and $\mu = (2)$. Then we start with

$$\begin{array}{|c|c|c|}\hline \star & \star & \\ \hline & & \\ \hline\end{array}$$

We get 3 excited diagrams:

$$\begin{array}{|c|c|c|}\hline \star & \star & \\ \hline & & \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline \star & & \\ \hline & & \star \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & \star & \star \\ \hline\end{array}$$

If we take only the first term of the sum, we get the following lower bound:

Corollary 1.4.

$$f^{\lambda\backslash\mu} \geq n! \prod_{(i,j)\in\lambda\backslash\mu} \frac{1}{h_{i,j}}$$